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**RESPONSE OF MORGAN COIL - PHASES**

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## Response of Morgan Coil - Phasor

### 1. Fields to be Measured.

We will characterize the field by a complex phasor function  $W = \gamma + i\phi = -\sum_{m=1} A_m (x+iy)^m = -\sum A_m R^m e^{im\theta}$ . The coefficient  $A_m$  is in general complex

$$A_m = R_m + i I_m = B_m e^{i\chi_m}$$

with  $R_m, I_m, B_m, \chi_m$  all real.  $\phi$  is the potential function  $\vec{B} = -\nabla \phi$ , and  $\gamma^*$  is the flux per unit length linking a line parallel to the  $z$  axis at  $(x, y)$  or  $(R, \theta)$ . To understand the sign conventions we note that if  $A$  is positive and real, the magnetic field  $B_R$  is positive at  $\theta = 0$  (normal multipole); if  $A$  is positive and imaginary,  $B_R$  is positive at  $\theta = 0$ . For example, for a pure normal quadrupole, the ray  $\theta = 0$  passes through the current bundle whose current is in the negative  $z$  direction [( $x_3$ ) right-handed].

The following relations hold

$$R_m = B_m \cos \chi_m \quad B_m = \sqrt{R_m^2 + I_m^2}$$

$$I_m = B_m \sin \chi_m \quad \chi_m = \arctan I_m / R_m$$

$\chi_m$  ( $I_m$ ) is the angle a general multipole must be rotated (in a negative sense) to make it normal. All the above relationships are equally true for fields integrated from  $-\infty < z < +\infty$ , whence  $\gamma$  becomes the total flux linking a line from  $-\infty < z < +\infty$  at  $R, \theta$ .

\* See Appendix.

## 2. Morgan Coil Response.

A morgan coil of order  $N$  has  $2N$  wires parallel to the  $z$  axis equally spaced on a cylinder of radius  $a$ , the axis of the cylinder being the  $z$  axis. Adjacent wires are connected such that they link flux in opposite directions. The angle of the coil is characterized by the angle of the wire which links flux positively\* and is brought out, with a neighboring wire, to the electronics. The voltage appearing across these leads is integrated to give the total flux linking the coil.

Then the total flux linking the morgan coil is

$$\begin{aligned} F(\theta) &= \sum_{j=0}^{2N-1} (-1)^j \left[ -\Phi(a, \theta + \frac{\pi j}{N}) \right] \\ &= \operatorname{Re} \sum_{j=0}^{2N-1} \sum_{m=1}^{\infty} B_m a^m e^{i(m\theta + \chi_m + \pi j + \frac{\pi j m}{N})} \end{aligned}$$

The sum over  $j$  is simple, yielding zero unless  $m = (2p-1)N$ ,  $p$  a positive integer, in which case it yields  $2N$ . Then

$$E(\theta) = \operatorname{Re} \sum_{m=(2p-1)N} B_m a^m 2N e^{i(m\theta + \chi_m)}$$

$$= \sum_{m=(2p-1)N} B_m a^m 2N \cos[m\theta + \chi_m]$$

\* This is the wire which proceeds from the positive rectangular terminal through the magnet in the negative  $z$  direction. Otherwise there is an overall sign shift.

### 3. Fourier Analysis.

Data is taken with an unknown starting point, and the integrated coil voltage is Fourier-analyzed. Let  $S$  be the starting point of the coil. Then  $\theta = \delta_N + \alpha$ , \* and the integrated coil output is ( $\text{Gain} = G/RC$ )

$$\begin{aligned} V_0 &= \sum V_n \cos(n\alpha + \phi_n) \\ &= -\frac{G}{RC} \sum_{m=(2p+1)N} B_m a^{m/2N} \cos[m\alpha + m\delta_N + \chi_m] \\ &= \frac{G}{RC} \sum_m B_m a^{m/2N} \cos[m\alpha + m\delta_N + \chi_m + \pi] \end{aligned}$$

Then  $V_n = \frac{G}{RC} B_n a^{n/2N}$

$$\phi_n = n\delta_N + \chi_m + \pi$$

Let  $M$  designate the magnet type (1=dipole 2=quad etc). Then the value of  $\chi_m$  is assumed to be zero. Alternatively we can say that the axis of all multipole is defined with respect to the principal multipole of the magnet. Then

$$\phi_M = M\delta_N + \pi, \text{ or } \delta_N = \frac{\phi_M}{M} - \frac{\pi}{M}$$

$$\text{and } \chi_n = \phi_n - \frac{n}{M} \phi_M + \frac{\pi n}{M} - \pi$$

\* The coil is assumed to rotate in the positive  $\theta$  direction  
† The (-) is due to Integrator Inversion.

## 4. Conventions

Several conventions were employed in reaching the preceding result. (1) Sign of integrator (2) sense of rotation of harmonic coil (3) sign, or phase of principal field component. If we adhere to (3) it is interesting to ask about the effect of 1 and 2 being reversed.

(1) Polarity reversal. This gives a change in signs in the relationship between  $F$  and  $V$  and leads to

$$X_n = \varphi_n - \frac{n}{M} \varphi_M$$

(2) Opposite Sense of Rotation. This changes the sign of  $\alpha$ , or equivalently  $\varphi_n$

$$X_n = -\varphi_n + \frac{n\varphi_M}{M} + \frac{n\pi}{M} - \pi$$

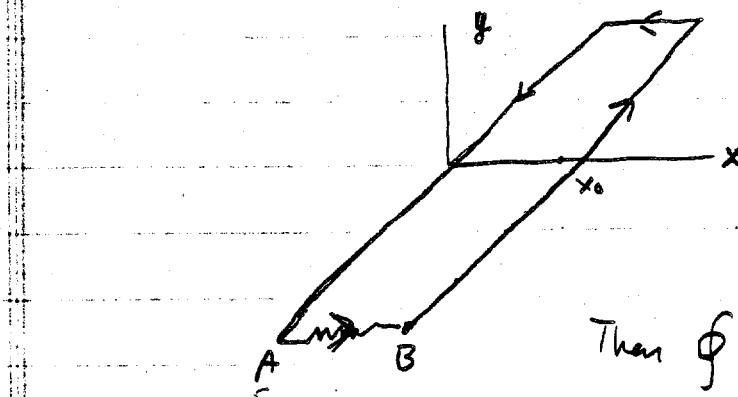
(1+2) Both

$$X_n = -\varphi_n + \frac{n\varphi_M}{M}$$

## Appendix

1)

Consider the loop, composed of  $y=x=0$ ;  $x=x_0, y=0$



$$\text{So } V_{in} = \frac{dF}{dt}$$

For positive  $B_y$ , the arrows give the positive circuit around the flux lines.

$$\text{Then } \oint E dl = -\frac{dF}{dt} = \int_A^B E dl = -V_B + V_A = -V_{in}$$

$$\text{Flux} = - \int V_{in} dt$$

$$V_o = - \frac{G}{RC} \int V_{in} dt$$

2) Comparison with Flux function

$$W = \gamma + i\varphi \quad B = -\nabla\varphi$$

$$F = \int_0^{x_0} B_y dx = \int dx \left( -\frac{\partial \varphi}{\partial y} \right)$$

$$= \int dx \left( -\frac{\partial \varphi}{\partial x} \right) = -(\varphi(x_0) - \varphi(0)) \quad \text{Then } -\varphi \text{ is the flux}$$

linking the loop above.

$$3) \sum_{z=0}^{2N-1} e^{i\pi j \left(1+\frac{m}{N}\right)} = \frac{1 - e^{i2N\pi \left(1+\frac{m}{N}\right)}}{1 - e^{i\pi \left(1+\frac{m}{N}\right)}} = \frac{1 - e^{2\pi i(N+m)}}{1 - e^{i\pi \left(1+\frac{m}{N}\right)}} = 0$$

unless  $e^{i\pi \left(1+\frac{m}{N}\right)} = 1$ , in which case

$$\sum = 1^0 + 1^1 + 1^2 + \dots + 1^{2N-1} = 2N$$

$$\text{If } e^{i\pi \left(1+\frac{m}{N}\right)} = 1 \text{ then } 1 + \frac{m}{N} = 2p, \text{ or } m = N(2p-1)$$

? any integer